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Low-energy limit of Yang–Mills with massless adjoint quarks: chiral Lagrangian and Skyrmions

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Abstract

If the fundamental quarks of QCD are replaced by massless adjoint quarks, the pattern of the chiral symmetry breaking drastically changes compared to the standard one. It becomes $SU(N_f) \rightarrow SO(N_f)$. While for $N_f = 2$, the chiral Lagrangian describing the ‘pion’ dynamics is well known, this is not the case at $N_f > 2$. We outline a general strategy for deriving chiral Lagrangians for the coset spaces $\mathcal{M}_k = SU(k)/SO(k)$ and study in detail the case of $N_f = k = 3$. We obtain two- and four-derivatives terms in the chiral Lagrangian on the coset space $\mathcal{M}_3 = SU(3)/SO(3)$, as well as the Wess–Zumino–Novikov–Witten term, in terms of an explicit parameterization of the quotient manifold. Then we discuss stable topological solitons supported by this Lagrangian. Aspects of relevant topological considerations scattered in the literature are reviewed. The same analysis applies to $SO(N)$ gauge theories with N_f Weyl flavours in the vector representation.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Recently, a nontrivial large- N equivalence between bosonic subsectors of different gauge theories has been established [1] (for a review see [2]). This planar equivalence connects, in particular, the Yang–Mills theory with N_f Dirac fermions in the two-index symmetric (or antisymmetric) representation of colour $SU(N)$ on the one side, with the theory with N_f Weyl quarks in the adjoint representation on the other side.

If the number of flavours $N_f > 1$, both theories under consideration have a chiral symmetry which is spontaneously broken. The pattern of the chiral symmetry breaking (χ SB) is different [3–5]. For N_f Dirac fermions in the two-index (anti)symmetric representation, the pattern of χ SB is identical to that of QCD, namely,

$$SU(N_f)_L \times SU(N_f)_R \rightarrow SU(N_f)_V. \quad (1)$$

On the other hand, in the $SU(N)$ gauge theories with N_f Weyl fermions in the adjoint representation we have the following χ SB pattern¹:

$$SU(N_f) \rightarrow SO(N_f). \quad (2)$$

Thus, in this case the low-energy effective theory is a sigma model on the target space

$$\mathcal{M}_k = SU(k)/SO(k) \quad (3)$$

with $k = N_f$. This effective theory describes the interactions of the Goldstone bosons of the theory, the ‘pions.’ (Let us note in passing that the same sigma model emerges in $SO(N)$ gauge theories with N_f Weyl fermions in the *vector* representation. In this case, for large enough N there is no upper bound on N_f .)

For two adjoint flavours, $\mathcal{M}_2 = SU(2)/SO(2) = S^2$. The corresponding sigma model is a well-studied $O(3)$ sigma model [6]. With a four-derivatives term included it goes under the name of the Skyrme–Faddeev model (or, sometimes, the Faddeev–Hopf model) [7]. Solitons in this model are intriguingly interesting because of their knotted structure. They are known as Hopf solitons and were extensively studied [7, 8] within the framework of a ‘glueball hypothesis’ [7] according to which the Hopf solitons may be relevant to the description of glueball states in pure Yang–Mills theory. The fact that they are certainly relevant in the studies of solitons built from pions was noted in [9] where a detailed analysis of the $N_f = 2$ case is presented. In application to chiral Lagrangians, it is natural to refer to these solitons as Hopf Skyrmions. At large N , the quasi-classical consideration of the Skyrmions² becomes theoretically justified [11, 12]; therefore, these solitons should be in one-to-one correspondence with certain hadronic states from the spectrum of the given microscopic theory (see [13] for a discussion of this problem in Yang–Mills with two-index (anti)symmetric matter).

As was mentioned, the two-flavour case is singled out by the fact that in this problem the effective low-energy Lagrangian is known, so that its analysis, as well as that of solitons it supports, can be carried out in more or less explicit manner, through a combination of analytic and numeric methods (see [9]). At the same time, to the best of our knowledge, sigma models on the target spaces (3) with $k = 3, 4$ and 5 have not been studied in the literature so far. In this work we fill the gap. First, we outline general considerations referring to three, four and five flavours. Then we derive, in an explicit form, the chiral Lagrangian for the sigma model on \mathcal{M}_3 . We discuss its features in much detail. In particular, we discuss solitons in this model, and how they match the Hopf Skyrmions of the $N_f = 2$ model if one ascribes a large mass term to the third flavour.

The topology of the target space gives us information about the solitons in the model. The second homotopy group, $\pi_2(\mathcal{M}_k)$, is relevant for the spectrum of the flux tubes³. On the other hand, $\pi_3(\mathcal{M}_k)$ gives us the spectrum of the particle-like solitons (Skyrmions). Moreover, $\pi_4(\mathcal{M}_k)$ and $\pi_5(\mathcal{M}_k)$ are relevant for the introduction of the Wess–Zumino–Novikov–Witten (WZNW) [15] term which, in the case of QCD, tells us how to quantize the Skyrmion, i.e. whether it becomes a fermion or a boson upon quantization [11, 12]. The relevant homotopy groups are shown in table 1.

¹ To ensure the very existence of the global chiral symmetry on the one hand, and to keep the microscopic theory asymptotically free on the other, we must assume that $2 \leq N_f \leq 5$. A more exact version of equation (3) is $SU(k) \times \mathbb{Z}_{2N} \rightarrow SO(k) \times \mathbb{Z}_2$ where the discrete factors are the remnants of the anomalous singlet axial $U(1)$; they play no role in what follows.

² The Skyrmions were introduced in particle physics long before QCD [10].

³ As was pointed out in [14], the Skyrme model with just the quadratic and quartic terms exhibits size instability for all vortices: under a spatial rescaling $r \rightarrow \lambda r$ the quadratic contribution stays invariant and the quartic one rescales by a factor λ^2 . Therefore, the energy is minimized at infinite size. As discussed in [14], this divergence can be eliminated, for example, by giving a bare mass to the quarks, which explicitly breaks the flavour symmetry and induces a potential term on the target space of the sigma model under consideration.

Table 1. Some homotopy groups for the manifolds \mathcal{M}_k (see [11, 16]). The relevant exact sequences are discussed in appendix A. The WZNW term cannot be introduced for $k = 2, 4$ because $\pi_4(\mathcal{M}_k)$ is nontrivial. On the other hand, the \mathbb{Z}_2 factors in $\pi_5(\mathcal{M}_{3,5})$ present no topological obstruction for the WZNW term.

k	$\dim \mathcal{M}_k$	π_2	π_3	π_4	π_5
2	2	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2
3	5	\mathbb{Z}_2	\mathbb{Z}_4	1	$\mathbb{Z} \otimes \mathbb{Z}_2$
4	9	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}	$\mathbb{Z} \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2$
5	14	\mathbb{Z}_2	\mathbb{Z}_2	1	$\mathbb{Z} \otimes \mathbb{Z}_2$
$k > 5$	$\frac{k^2+k-2}{2}$	\mathbb{Z}_2	\mathbb{Z}_2	1	\mathbb{Z}

For $k = 2$, the target space \mathcal{M}_2 reduces to a two-dimensional sphere S^2 . The corresponding sigma model supports flux tubes due to the fact that $\pi_2(\mathcal{M}_2) = \mathbb{Z}$, which are classified by integer topological numbers.

Since $\pi_3(\mathcal{M}_2) = \mathbb{Z}$, the Hopf Skyrmions are also classified by integers (these solitons can be understood as twisted flux tubes; mathematically, this can be shown by using the Hopf fibration, which gives us the first topologically nontrivial map between S^3 and S^2).

Furthermore, we have $\pi_4(S^2) = \mathbb{Z}_2$, implying that it is possible to quantize the Hopf Skyrmions both as bosons or as fermions [17–19]. There is no WZNW term for two flavours. In order to have the WZNW term, the target manifold of the sigma model in question must have dimension five or larger.

We will dwell on the $N_f = 3$ case. We will introduce an explicit parameterization of the coset space $\mathcal{M}_3 = SU(3)/SO(3)$ and construct an explicit Lagrangian for this sigma model, including the quadratic and quartic terms. It is shown that the homotopy class relevant for the \mathbb{Z}_2 vortices and for the \mathbb{Z}_4 Skyrmions supported by this Lagrangian can be obtained by an embedding of the corresponding homotopy class from \mathcal{M}_2 . The WZNW term will be calculated. We will show that it is proportional to the 5-volume form on \mathcal{M}_3 .

The organization of the paper is as follows. In section 2, we outline a general formalism allowing one to construct sigma models on G/H manifolds. We review the application of this formalism to the $SU(2)/SO(2)$ case; as a warm-up exercise we derive in this formalism the chiral Lagrangian of the $O(3)$ Skyrme–Faddeev model. In section 3, we apply it to the $k = 3$ case. We introduce explicit coordinates and then obtain the metric on $\mathcal{M}_3 = SU(3)/SO(3)$. The two-derivative part of the Lagrangian is presented in section 3, while the four-derivative part in appendix B. Topological aspects relevant to various solitonic configurations in the $SU(3)/SO(3)$ sigma model are discussed in section 4. The WZNW term on \mathcal{M}_3 is calculated in section 5. Appendix A presents the exact sequences for some homotopy groups used in the paper.

2. General considerations

To refresh memory, it is convenient to start from the well-known case of QCD with N_f Dirac quarks. Then the Lagrangian of the Skyrme model includes the following two- and four-derivatives terms:

$$\begin{aligned}
 \mathcal{L} &= \frac{F_\pi^2}{4} \mathcal{L}_2 + \frac{1}{e^2} \mathcal{L}_4 \\
 &= \frac{F_\pi^2}{4} \text{Tr}(\partial_\mu U \partial^\mu U^\dagger) + \frac{1}{e^2} \text{Tr}[(\partial_\mu U)U^\dagger, (\partial_\nu U)U^\dagger]^2,
 \end{aligned}
 \tag{4}$$

where the matrix U is an element of the $SU(N_f)$ group, and F_π and e are constants. The subscript π will be omitted hereafter. The two-derivatives term is just the kinetic term of the Goldstone bosons of the theory; mathematically, it is the metric of the target manifold. The four-derivatives term is needed in order to stabilize the particle-like solutions, which otherwise would tend to shrink to zero size. The coset space corresponding to (1) is a groups space itself.

In the generic case of the group quotient G/H , a general prescription for obtaining two-derivatives terms was given long ago in [20]. This issue has been recently discussed anew in [21] in a slightly modified perspective pertinent to the Faddeev–Skyrme models. Following the formalism of the latter paper, we get for the two-derivatives term

$$\mathcal{L}_2 = \text{Tr}(P_{h\perp}(U^\dagger \partial_\mu U) \cdot P_{h\perp}(U^\dagger \partial_\mu U)), \quad (5)$$

where $P_{h\perp}$ is the projection in the Lie algebra of G on the space orthogonal to the Lie algebra of H , which we call h . The construction of $P_{h\perp}$ will be discussed momentarily. Analogously, the four derivatives term is

$$\mathcal{L}_4 = \text{Tr}[P_{h\perp}(U^\dagger \partial_\mu U), P_{h\perp}(U^\dagger \partial_\nu U)]^2. \quad (6)$$

As a warm-up exercise let us discuss first the Faddeev–Skyrme model, in which $G = SU(2)$ and $H = SO(2)$, and the explicit form of the Lagrangian is well known. The quotient can be parameterized using the matrix exponential of the $SU(2)$ generators which are not in the chosen $H = SO(2) = U(1)$. Let us assume that the $U(1)$ factor is generated by the second Pauli matrix,

$$e^{i\sigma_2 t}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \quad (7)$$

Then these generators of G/H are the symmetric self-adjoint two-by-two matrices. Any such element can be parameterized as

$$U = \exp(iV \cdot A \cdot V^\dagger), \quad (8)$$

where A is the diagonal matrix,

$$A = \begin{pmatrix} +\theta/2 & 0 \\ 0 & -\theta/2 \end{pmatrix}, \quad (9)$$

and

$$V = \begin{pmatrix} \cos \alpha/2 & -\sin \alpha/2 \\ \sin \alpha/2 & \cos \alpha/2 \end{pmatrix}. \quad (10)$$

With this parameterization we recover the standard S^2 , provided

$$0 \leq \theta \leq \pi, \quad 0 \leq \alpha \leq 2\pi. \quad (11)$$

Indeed, the projection $P_{h\perp}(T)$ defined on the Lie algebra of $SU(2)$ is given by

$$P_{h\perp}(T) = T - \frac{1}{2}\sigma_2 \text{Tr}(T \cdot \sigma_2). \quad (12)$$

Then we obtain equation (8) and, from equation (5), we arrive at the two-derivatives term presenting the standard on S^2 ,

$$\frac{1}{2}[(\partial_\mu \theta)^2 + \sin^2 \theta (\partial_\mu \alpha)^2] = \frac{1}{2}(\partial_\mu \vec{n})^2, \quad \vec{n} \cdot \vec{n} = 1. \quad (13)$$

Furthermore, the four-derivatives term is recovered from equation (6),

$$\frac{1}{2} \sin^2 \theta (\partial_\mu \theta \partial_\nu \alpha - \partial_\nu \theta \partial_\mu \alpha)^2, \quad (14)$$

which identically reduces to

$$\frac{1}{2}(\partial_\mu \vec{n} \wedge \partial_\nu \vec{n})^2. \quad (15)$$

3. An explicit Lagrangian for $k = 3$

We can proceed in a way analogous to what we have just done for the $k = 2$ case. We parameterize the quotient using the matrix exponential of the generators of $SU(3)$ which are not in $SO(3)$. These generators are the symmetric 3×3 matrices. It is always possible to diagonalize a symmetric matrix in an orthogonal basis. We introduce the parameters θ, η for the eigenvalues of the matrix and the parameters α, β, γ as the Euler angles for the transformation which brings the generic symmetric matrix in diagonal form. The angular range of each of the five parameters mentioned above will be determined using the $SO(3)$ quotient relations.

The parameterization we use is as follows:

$$U = \exp(iV \cdot A \cdot V^\dagger) \tag{16}$$

where

$$A = \frac{1}{2} \begin{pmatrix} \eta/\sqrt{3} + \theta & 0 & 0 \\ 0 & \eta/\sqrt{3} - \theta & 0 \\ 0 & 0 & -2\eta/\sqrt{3} \end{pmatrix} \tag{17}$$

and V is an $SO(3)$ matrix parameterized by three Euler angles α, β, γ ,

$$V = \begin{pmatrix} \cos \frac{\alpha}{2} \cos \frac{\gamma}{2} - \cos \frac{\beta}{2} \sin \frac{\alpha}{2} \sin \frac{\gamma}{2} & -\sin \frac{\alpha}{2} \cos \frac{\gamma}{2} - \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \sin \frac{\gamma}{2} & \sin \frac{\beta}{2} \sin \frac{\gamma}{2} \\ \cos \frac{\alpha}{2} \sin \frac{\gamma}{2} + \cos \frac{\beta}{2} \sin \frac{\alpha}{2} \cos \frac{\gamma}{2} & -\sin \frac{\alpha}{2} \sin \frac{\gamma}{2} + \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2} & -\cos \frac{\gamma}{2} \sin \frac{\beta}{2} \\ \sin \frac{\alpha}{2} \sin \frac{\beta}{2} & \cos \frac{\alpha}{2} \sin \frac{\beta}{2} & \cos \frac{\beta}{2} \end{pmatrix}. \tag{18}$$

The angle variation range for the θ is

$$0 \leq \theta \leq \pi. \tag{19}$$

The range for θ comes from the following equivalence which holds modulo $SO(3)$ conjugation:

$$\theta \rightarrow 2\pi - \theta, \quad \alpha \rightarrow \alpha \pm \pi. \tag{20}$$

In other words,

$$U_{\theta, \alpha} \cdot (U_{2\pi - \theta, \alpha \pm \pi})^{-1} \in SO(3). \tag{21}$$

In addition, the action of α rotations modulo $SO(3)$ is trivial at $\theta = \pi$. The range for η is

$$-\frac{\theta}{\sqrt{3}} \leq \eta \leq \frac{\theta}{\sqrt{3}}. \tag{22}$$

This is due to the fact that we do not have to double-count different eigenvalue orderings (we can make an arbitrary permutation of the diagonal elements by applying a combination of $\alpha = \pi, \beta = \pi$ and $\gamma = \pi$ rotations). At $\eta = \pm\theta/\sqrt{3}$ we observe that two of the three elements are degenerate.

Finally, the range of variation for the Euler parameters is

$$0 \leq \alpha \leq 2\pi, \quad 0 \leq \beta \leq 2\pi, \quad 0 \leq \gamma \leq 2\pi. \tag{23}$$

These ranges come from the following three distinct invariances for the matrix U :

$$\begin{aligned} \alpha &\rightarrow \alpha + 2\pi, \\ \alpha &\rightarrow 2\pi - \alpha, \quad \beta \rightarrow \beta + 2\pi, \\ \alpha &\rightarrow 2\pi - \alpha, \quad \beta \rightarrow 2\pi - \beta, \quad \gamma \rightarrow \gamma + 2\pi. \end{aligned} \tag{24}$$

The Lie algebra of $H = SO(3)$ is generated by three Gell–Mann matrices, λ_2 , λ_5 and λ_7 ,

$$\lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}. \quad (25)$$

The projector $P_{h\perp}(T)$ is

$$P_{h\perp}(T) = T - \frac{1}{2}\{\lambda_2\text{Tr}(T \cdot \lambda_2) + \lambda_5\text{Tr}(T \cdot \lambda_5) + \lambda_7\text{Tr}(T \cdot \lambda_7)\}. \quad (26)$$

The two-derivatives term can be obtained upon substituting the parameterization (16)–(18) into the general equation (5),

$$\begin{aligned} \mathcal{L}_2 = \frac{1}{4} & \left[2(\partial_\mu\theta)^2 + 2(\partial_\mu\eta)^2 + 2\sin^2\theta(\partial_\mu\alpha)^2 \right. \\ & + (1 - \cos\sqrt{3}\eta\cos\theta - \cos\alpha\sin\sqrt{3}\eta\sin\theta)(\partial_\mu\beta)^2 \\ & + \frac{1}{2}(\partial_\mu\gamma)^2(2 - (1 + \cos\beta)\cos^2\theta - 2\cos\sqrt{3}\eta\cos\theta\sin^2\frac{\beta}{2} \\ & + 2\cos\alpha\sin^2\frac{\beta}{2}\sin\sqrt{3}\eta\sin\theta + \sin^2\theta + \cos\beta\sin^2\theta) \\ & + \left(4\cos\frac{\beta}{2}\sin^2\theta\right)(\partial_\mu\alpha)(\partial_\mu\gamma) \\ & \left. - \left(2\sin\alpha\sin\frac{\beta}{2}\sin\sqrt{3}\eta\sin\theta\right)(\partial_\mu\beta)(\partial_\mu\gamma) \right]. \quad (27) \end{aligned}$$

As a nontrivial check we can compute from this metric the scalar curvature. We find that it is constant as is required for the symmetric space,

$$r = 15. \quad (28)$$

Moreover, the Ricci tensor is proportional to the metric (\mathcal{M}_3 is an Einstein manifold, as for many other coset spaces),

$$R_{ab} = 3g_{ab}, \quad (29)$$

where g_{ab} is just the metric in equation (27) written in the tensorial form.

The four-derivatives term can be computed from equation (6); the result is quite bulky. The explicit expression for the four-derivatives term is given in appendix B.

4. Topology and solitons

4.1. Topology of the sections at constant (θ, η)

Let us discuss figure 1 in some detail. For every fixed value of (θ, η) we have a submanifold $\mathcal{R}(\theta, \eta)$. First of all let us consider the topology of $\mathcal{R}(\theta, \eta)$ for a generic value inside the triangle in figure 1,

$$-\theta/\sqrt{3} < \eta < \theta/\sqrt{3} \quad \text{and} \quad 0 < \theta < \pi.$$

Each of these submanifolds is parameterized by a generic $SO(3)$ rotation with the Euler angles α, β, γ .

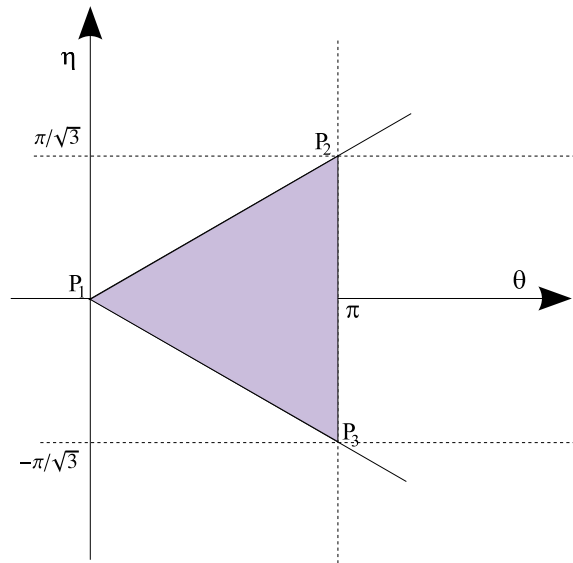


Figure 1. Range of variation for θ and η . At the triangle vertices the action of the Euler rotations is trivial (they correspond to points on \mathcal{M}_3). These three points are in correspondence with the centre \mathbb{Z}_3 of $SU(3)$. Any point on the sides of the triangle corresponds to a two-dimensional submanifold, while any point in the interior corresponds to a three-dimensional submanifold.

There is a subtle point, however. Some of these $SO(3)$ elements have a trivial action. These element constitute a $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ subgroup of $SO(3)$, let us call it \mathcal{A} ,

$$\mathcal{A} = \{1, a, b, a \cdot b\}, \tag{30}$$

where

$$a = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad a \cdot b = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{31}$$

It is not difficult to check that

$$a^2 = b^2 = (a \cdot b)^2 = 1. \tag{32}$$

From expressions above it is rather obvious that \mathcal{A} is a subgroup of $SO(3)$.

It is well known that $SU(2)$ and $SO(3)$ differ by the centre element \mathbb{Z}_2 ,

$$SO(3) = SU(2)/\mathbb{Z}_2. \tag{33}$$

It is convenient to introduce the projection operator \mathcal{P} ,

$$SU(2) \xrightarrow{\mathcal{P}} SO(3). \tag{34}$$

Now, we will need to build an eight-element subgroup $\tilde{\mathcal{A}}$ of $SU(2)$ which is in the same relation to \mathcal{A} as in equation (34), namely,

$$\tilde{\mathcal{A}} = \mathcal{P}^{-1}\mathcal{A}. \tag{35}$$

The eight elements of the subgroup $\tilde{\mathcal{A}}$ are as follows: let us call the \mathbb{Z}_2 centre element in equation (33) as \tilde{c} . Moreover,

$$\tilde{a} = \exp(i\pi\sigma_3/2), \quad \tilde{b} = \exp(i\pi\sigma_1/2). \tag{36}$$

Then

$$\tilde{\mathcal{A}} = \{1, \tilde{a}, \tilde{a}^2 = \tilde{c}, \tilde{a}^3, \tilde{a}\tilde{b}, \tilde{a}^2\tilde{b}, \tilde{a}^3\tilde{b}, \tilde{a}^4\tilde{b} = \tilde{b}\}. \quad (37)$$

This is a subgroup of $SU(2)$ isomorphic to the dihedral group D_4 ,

$$\tilde{\mathcal{A}} \sim D_4.$$

The group D_4 has three possible \mathbb{Z}_4 subgroups, each of them generated by powers of \tilde{a} , \tilde{b} , $\tilde{a}\tilde{b}$. The conclusion we arrive at is

$$\mathcal{R}(\theta, \eta) = SU(2)/D_4, \quad (38)$$

which entails

$$\pi_1(\mathcal{R}(\theta, \eta)) = D_4. \quad (39)$$

Equation (39) is due to the fact that $SU(2)$ is simply connected.

Now let us consider ‘degenerate’ values of (η, θ) on a side of the triangle ($\eta = \pm\theta/\sqrt{3}$ or $\theta = \pi$) in figure 1. In such points we have that the $SO(3)$ group degenerates into $SO(2) \times \mathbb{Z}_2$. For example, at $\eta = -\theta/\sqrt{3}$, the $SO(2)$ subgroup is generated by

$$\exp \left\{ i \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} t \right\}, \quad (40)$$

and the \mathbb{Z}_2 element is

$$b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (41)$$

We conclude that on the three segments $\eta = \theta/\sqrt{3}$, $\eta = -\theta/\sqrt{3}$ and $\theta = \pi$

$$\mathcal{R}(\theta, \eta) = \frac{SO(3)/SO(2)}{\mathbb{Z}_2} = S^2/\mathbb{Z}_2 = \mathbb{P}\mathbb{R}^2, \quad (42)$$

which implies, of course,

$$\pi_1(\mathcal{R}(\theta, \eta)) = \mathbb{Z}_2. \quad (43)$$

If, in consideration of the $\mathcal{R}(\theta, \eta)$ section we continuously move from a point in the internal part of the triangle to a point on one of its three sides, we have that a \mathbb{Z}_4 subgroup of the fundamental group D_4 becomes trivial. We have that a different \mathbb{Z}_4 subgroup becomes trivial on each of the sides of the triangle, namely,

$$\begin{aligned} \eta = \theta/\sqrt{3} &\rightarrow (1, \tilde{b}, \tilde{b}^2, \tilde{b}^3), \\ \eta = -\theta/\sqrt{3} &\rightarrow (1, \tilde{a}\tilde{b}, (\tilde{a}\tilde{b})^2, (\tilde{a}\tilde{b})^3), \\ \theta = \pi &\rightarrow (1, \tilde{a}, \tilde{a}^2, \tilde{a}^3). \end{aligned} \quad (44)$$

Finally, if we consider the vertices $P_{1,2,3}$ of the triangle, the action of the Euler rotations modulo the unbroken $SO(3)$ is trivial. Therefore, in correspondence with these three values, we find that $\mathcal{R}(P_{1,2,3})$ is a point.

4.2. Homotopy group generators

After this discussion we are ready to elucidate how to explicitly build the 2- and 3-cycles in our parameterization. The vortex soliton will wrap on a nontrivial 2-cycle while the Skyrmion will wrap on a 3-cycle, so this discussion is important for understanding of how to build the solitons in the theory at hand.

Let us start with the 2-cycle. From [11] we know that

$$\pi_2(\mathcal{M}_3) = \mathbb{Z}_2.$$

Hence, the problem is to identify the homotopy class of the only topologically nontrivial map from S^2 onto \mathcal{M}_3 .

Let us denote (θ_s, ϕ_s) the standard coordinates on S^2 . We then can build this nontrivial map in the following way: we map the north pole of S^2 onto P_1 and the south pole onto P_2 . We map the θ_s coordinate of the sphere along the $\eta = -\theta/\sqrt{3}$ line, with the relation $\theta_s = \theta$. The ϕ_s coordinate, on the other hand, is mapped continuously onto a representative of the nontrivial 1-cycle of

$$\pi_1(\mathcal{R}(\theta, \eta = -\theta/\sqrt{3})) = \mathbb{Z}_2.$$

For example, $\beta, \gamma = 0$ and $\alpha = \phi_s$.

There is no way to shrink this map to a point. It is possible, say, to continuously transform the map from the segment P_1P_2 to P_1P_3 or to P_2P_3 , but it is impossible to shrink the map to trivial in this way. Also, if we compose this map twice, as in the definition of π_2 , we find a topologically trivial map.

This map is also homotopic to the map

$$\theta = \theta_s, \quad \alpha = \phi_s,$$

with $\eta = \beta = \gamma = 0$. The image of this map is in the \mathcal{M}_2 submanifold of \mathcal{M}_3 defined by the constraint $\eta = \beta = \gamma = 0$; the homotopy class corresponds to the vortex with the minimal winding in \mathcal{M}_2 . This shows that if we embed the minimal winding vortex of the Faddeev–Skyrme model in \mathcal{M}_3 , we obtain a representative of the homotopy class of the \mathbb{Z}_2 vortex. On the other hand, the vortices with nonminimal winding are unstable if embedded in \mathcal{M}_3 : those with the even winding number will decay to the topologically trivial configuration and those with the odd winding number to the \mathbb{Z}_2 minimal vortex.

From the exact sequence of the homotopy group of a fibre bundle (discussed in appendix A), we know not only that

$$\pi_3(\mathcal{M}_3) = \mathbb{Z}_4,$$

but, in addition, that the elements of \mathbb{Z}_4 are the projection modulo 4 of

$$\pi_3(SU(3)) = \mathbb{Z}$$

induced by the quotient procedure. In other words, if we take a homotopy class $n \in \pi_3(SU(3))$ it corresponds to the n modulo 4 class in $\pi_3(\mathcal{M}_3)$.

We also know that the elements of $\pi_3(SU(3))$ are just those of the embedded $\pi_3(SU(2))$. The projection induced by the Hopf fibration gives a one-to-one correspondence between

$$\pi_3(SU(2)) = \mathbb{Z} \quad \text{and} \quad \pi_3(\mathcal{M}_2 = SU(2)/SO(2)) = \mathbb{Z}.$$

This tells us that if we embed the solutions of the Faddeev–Skyrme model in \mathcal{M}_3 , they are topologically stable modulo 4. Thus, the solutions with the Hopf number $4n$ are topologically trivial in \mathcal{M}_3 , while the others will tend to decay to the minimal \mathbb{Z}_4 representative. This gives us an upper bound on the mass of each of the three \mathbb{Z}_4 Skyrmions from the mass of the

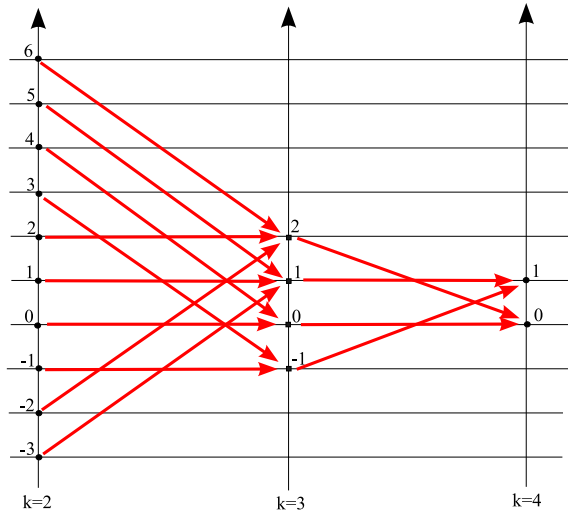


Figure 2. The Skyrmions of the theory with $k = 2$ (the Faddeev–Skyrme model) are labelled by integer $n \in \mathbb{Z}$. If embedded in the theory with $k = 3$, they will tend to decay to the corresponding \mathbb{Z}_4 topological class. If we further embed the Skyrmions in the theory with $k \geq 4$ only the Skyrmions with \mathbb{Z}_2 topological class will survive.

corresponding Skyrmion in the Faddeev–Skyrme model (the three relevant ones are those with the Hopf numbers 1, 2 and -1).

An interesting problem is to study the explicit breaking of the $SU(3)$ flavour symmetry in \mathcal{M}_3 . To this end, one can introduce a mass term $m_3 \neq 0$ to the quark of the third flavour in the microscopic theory. This mass term breaks the flavour group $SU(3)$ down to $SU(2)$. In the low-energy effective theory, with the chiral Lagrangian (27), a potential term on \mathcal{M}_3 will be generated (which will vanish, of course, on the \mathcal{M}_2 submanifold). If $m_3 \rightarrow \infty$ all Skyrmion maps are stable since $\pi_3(\mathcal{M}_2) = \mathbb{Z}$.

At finite m_3 , the Skyrmions with the winding number larger than 2 and smaller than -1 will become metastable. They will tunnel to the four stable configurations (see figure 2). If m_3 is large enough, it should be possible to calculate the lifetimes of the metastable states by using semiclassical methods.

If we further embed the model in \mathcal{M}_k with $k \geq 4$, some of the \mathbb{Z}_4 Skyrmions will become unstable and will decay into the \mathbb{Z}_2 Skyrmions.

5. Wess–Zumino–Novikov–Witten term

If $\pi_4(G/H)$ is trivial there is no topological obstruction for introduction of the WZNW term⁴. This condition is satisfied for $k = 3$ and for $k \geq 5$. We can naturally generalize the expression from the one referring to the $SU(N)$ case, discussed in [11, 12],

$$S_{\text{WZNW}} \propto \int_{B_5} d\Sigma^{\mu\nu\rho\sigma\lambda} \text{Tr}\{P_{h\perp}(U^\dagger \partial_\mu U) \cdot P_{h\perp}(U^\dagger \partial_\nu U) \times P_{h\perp}(U^\dagger \partial_\rho U) \cdot P_{h\perp}(U^\dagger \partial_\sigma U) \cdot P_{h\perp}(U^\dagger \partial_\lambda U)\}. \tag{45}$$

⁴ Ideas as to how one could introduce a nonstandard WZNW term in the cases of nontrivial $\pi_4(G/H)$ are discussed in [22].

In order to avoid an ambiguity in the quantization procedure due to different possible choices of B_5 for a given S^4 boundary (see [11]), we have to require the contribution of this term to be a multiple of 2π if integrated on an arbitrary S^5 manifold. The integral of this term over the S^5 manifold is a topological invariant which depends on the topological class in $\pi_5(G/H)$. The value of the integral vanishes for the S^5 cycles in finite cyclic factors \mathbb{Z}_k of $\pi_5(G/H)$ (there are indeed \mathbb{Z}_2 factors in $\pi_5(\mathcal{M}_k)$ for $k = 3, 5$, but they are irrelevant for the WZNW term). On the other hand, the integral can be non-vanishing on the \mathbb{Z} factor, and its value is proportional to the ‘winding number.’

We have to normalize S_{WZNW} as follows:

$$S_{\text{WZNW}} = nA \int_{B_5} d\Sigma^{\mu\nu\rho\sigma\lambda} \text{Tr}\{P_{h\perp}(U^\dagger \partial_\mu U) \cdot P_{h\perp}(U^\dagger \partial_\nu U) \times P_{h\perp}(U^\dagger \partial_\rho U) \cdot P_{h\perp}(U^\dagger \partial_\sigma U) \cdot P_{h\perp}(U^\dagger \partial_\lambda U)\}, \tag{46}$$

where the normalization factor A is chosen in such a way that the integral on the map with the minimal winding (in the \mathbb{Z} factor of the π_5) between S^5 and G/H is 2π and n is an arbitrary integer.

In the case of \mathcal{M}_3 , we calculated the WZNW term using the parameterization introduced in section 3. It is proportional to the volume form of the manifold (this is due to the fact that our target manifold is five dimensional). Namely,

$$S_{\text{WZNW}} = nA \frac{i60}{64\sqrt{3}} \int_{B_5} d\Sigma^{\mu\nu\rho\sigma\lambda} (\partial_\mu \theta \cdot \partial_\nu \eta \cdot \partial_\rho \alpha \cdot \partial_\sigma \beta \cdot \partial_\lambda \gamma) \times (\cos \sqrt{3}\eta - \cos \theta) \sin \frac{\beta}{2} \sin \theta. \tag{47}$$

The coefficient A must be adjusted to make the integral 2π on the map from S_5 to \mathcal{M}_3 corresponding to the minimal winding. The element of

$$\pi_5(\mathcal{M}_3) = \mathbb{Z} \times \mathbb{Z}_2$$

with the minimal winding in the \mathbb{Z} factor makes $l = 2$ windings around the manifold. As a result, we find the following value for the normalization factor A :

$$A = -\frac{2i}{15\pi^2}. \tag{48}$$

6. Conclusions and outlook

If the fundamental quarks of QCD are replaced by massless adjoint quarks, the pattern of the chiral symmetry breaking is $SU(N_f) \rightarrow SO(N_f)$. This work addresses and solves the issue of constructing sigma models on the coset spaces $SU(N_f)/SO(N_f)$. The only case which had been explicitly solved previously is $N_f = 2$. This is the celebrated $O(3)$ or $CP(1)$ model. We focused mainly on $N_f = 3$, presenting a full solution in this particular case, with a few general remarks on $N_f > 3$ scattered in the bulk of the paper. These remarks outline a general strategy for constructing the $SU(N_f)/SO(N_f)$ sigma models for arbitrary N_f .

We found an explicit parameterization for the sigma model with the target space $\mathcal{M}_3 = SU(3)/SO(3)$ in terms of five angles. The low-energy effective chiral Lagrangian is presented in equations (27), (47) and (B.2). As a check we computed the scalar curvature for the metric we got, and we found a constant, as is required for any homogeneous space.

We obtained WZNW term too. Due to the fact \mathcal{M}_3 is a five-dimensional manifold, the WZNW term is proportional to the volume form.

We discussed the topological side of the $SU(3)/SO(3)$ sigma models. The nontrivial homotopy classes of $\pi_2(\mathcal{M}_3) = \mathbb{Z}_2$ and $\pi_3(\mathcal{M}_3) = \mathbb{Z}_4$, relevant for the vortex lines and Skyrmions, can be found by embedding in \mathcal{M}_3 some nontrivial homotopy classes of the Faddeev–Skyrme model.

We can say that the algebraic aspect of the low-energy chiral dynamics corresponding to the χ SB pattern (2) is in essence clear at the moment. This problem has another aspect, dynamical, related to interpreting the algebraic results obtained above in the language of the underlying microscopic theory—Yang–Mills with the adjoint quarks. Since $\pi_3(\mathcal{M}_k)$ is nontrivial, the $SU(N_f)/SO(N_f)$ chiral Lagrangians predict some ultraheavy stable solitons, analogues of the QCD Skyrmions, whose mass scales as N_c^2 at large N_c . The question is can we understand these solitons (and their stability) in the language of the microscopic (ultraviolet) theory?

This question obviously should be addressed and answered in the framework of an independent project whose thrust is on dynamical roots of the soliton stability in the Yang–Mills theory with the adjoint quarks. The work in this direction has just started, with first results reported in a follow-up publication [9].

In conventional QCD, the Skyrme topological charge is matched with the baryon number; in this way, the Skyrmions can be identified with baryons, and their stability is protected by the global symmetry—the baryon charge conservation.

In adjoint QCD there is no such obvious reason for stability; the analogue of the baryon charge, the fermion number, is broken first to $\mathbb{Z}_{2N_c N_f}$ by the chiral anomaly; this discrete symmetry is then spontaneously broken to \mathbb{Z}_2 by the fermion condensates. This \mathbb{Z}_2 symmetry is not sufficient by itself to protect the soliton from decaying. This is due to the fact that in addition to the Goldstone bosons, which of course have vanishing fermion number, we expect light composite fermions of the form

$$\psi_{\beta f} \propto \text{Tr}(\lambda_f^\alpha \sigma_{\alpha\beta}^{\mu\nu} F_{\mu\nu}), \quad (49)$$

with an odd fermion number (in this expression λ_f^α is the adjoint Weyl fermion and $\sigma_{\alpha\beta}^{\mu\nu} F_{\mu\nu}$ is the gluon field strength field in the spinorial notation).

The problem of the soliton stability in the case $N_f = 2$ is solved in [9]. The solitons turn out to be in correspondence with exotic hadrons with mass $O(N_c^2)$ and $P = (-1)^Q (-1)^F = -1$, where Q is the conserved charge corresponding to the unbroken $U(1)$ flavour subgroup. All other lighter degrees of freedom have $P = 1$; the Goldstone bosons have zero fermion number and even Q charge; the light fermions ψ have an odd Q charge and odd fermion number. This is just a \mathbb{Z}_2 stability (a configuration with the Hopf number two can indeed decay to an array of π 's and ψ 's). To detect this phenomenon in the low-energy chiral theory we need to introduce the fermions ψ in the effective low-energy sigma model.

This problem for $N_f > 2$ is currently under investigation.

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Appendix A. Exact sequences for some homotopy groups

A.1. π_2

The $k = 2$ case is special,

$$\cdots \rightarrow \pi_2(SU(k)) \rightarrow \pi_2(SU(2)/SO(2)) \rightarrow \pi_1(SO(2)) \rightarrow \pi_1(SU(2)) \rightarrow \cdots$$

$$\cdots \rightarrow 0 \rightarrow \mathbb{X} \rightarrow \mathbb{Z} \rightarrow 0 \rightarrow \cdots,$$

which gives us $\mathbb{X} = \mathbb{Z}$.

For $k > 2$ we have the following exact sequence:

$$\cdots \rightarrow \pi_2(SU(k)) \rightarrow \pi_2(SU(k)/SO(k)) \rightarrow \pi_1(SO(k)) \rightarrow \pi_1(SU(k)) \rightarrow \cdots$$

$$\cdots \rightarrow 0 \rightarrow \mathbb{X} \rightarrow \mathbb{Z}_2 \rightarrow 0 \rightarrow \cdots,$$

which gives us $\mathbb{X} = \mathbb{Z}_2$.

A.2. π_3

For $k = 2$ we know that the result is given by the Hopf fibration, $\pi_3(S^2) = \mathbb{Z}$.

For $k = 3$ and $k \geq 5$ we have the following exact sequence:

$$\cdots \rightarrow \pi_3(SO(k)) \rightarrow \pi_3(SU(k)) \rightarrow \pi_3(SU(k)/SO(k)) \rightarrow \pi_2(SO(k)) \rightarrow \cdots$$

$$\cdots \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{X} \rightarrow 0 \rightarrow \cdots,$$

which gives us $\mathbb{X} = \mathbb{Z}_s$ where s is the rank of the map between $\pi_3(SO(k))$ and $\pi_3(SU(k))$ induced by the embedding $SO(k) \rightarrow SU(k)$.

The number s can be calculated using the ‘winding number’ integral discussed in [23, 24],

$$s = -\frac{1}{24\pi^2} \int_{S^3} \text{Tr}(U^\dagger dU)^3, \tag{A.1}$$

where this integral is calculated on the smaller non-zero element of $\pi_3(SO(k))$. For $SO(3)$ a representative of the minimal element of π_3 is

$$(\theta, \phi, \rho) \rightarrow \exp(iq_j \hat{n}_j \rho), \tag{A.2}$$

where S^3 is parameterized by

$$\hat{n}_j = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta), \quad 0 < \rho < 2\pi,$$

and

$$q_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}, \quad q_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad q_3 = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For $k \geq 5$, we have to use

$$q_1 = 1/2 \begin{pmatrix} 0 & 0 & i & 0 & \cdots \\ 0 & 0 & 0 & -i & \cdots \\ -i & 0 & 0 & 0 & \cdots \\ 0 & i & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad q_2 = 1/2 \begin{pmatrix} 0 & 0 & 0 & i & \cdots \\ 0 & 0 & i & 0 & \cdots \\ 0 & -i & 0 & 0 & \cdots \\ -i & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$q_3 = 1/2 \begin{pmatrix} 0 & i & 0 & 0 & \cdots \\ -i & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & i & \cdots \\ 0 & 0 & -i & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where the dots denote zeros. This gives $s = 4$ for $k = 3$ and $s = 2$ for $k \geq 5$.

The $k = 4$ case is particular,

$$\begin{aligned} \cdots \rightarrow \pi_3(SO(4)) \rightarrow \pi_3(SU(4)) \rightarrow \pi_3(SU(4)/SO(4)) \rightarrow \pi_2(SO(4)) \rightarrow \cdots \\ \cdots \rightarrow \mathbb{Z} \otimes \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{X} \rightarrow 0 \rightarrow \cdots. \end{aligned}$$

Again the elements of $\pi_3(SU(4)/SO(4))$ are in correspondence with the elements of $\pi_3(SU(4))$ which are not homotopic to any elements of $\pi_3(SO(4))$. The same winding number argument used in the previous case for $k \geq 5$ gives us $\mathbb{X} = \mathbb{Z}_2$.

A.3. π_4

The $k = 2$ case is singled out,

$$\begin{aligned} \cdots \rightarrow \pi_4(SO(2)) \rightarrow \pi_4(SU(2)) \rightarrow \pi_4(SU(2)/SO(2)) \rightarrow \pi_3(SO(2)) \rightarrow \cdots \\ \cdots \rightarrow 0 \rightarrow \mathbb{Z}_2 \rightarrow \mathbb{X} \rightarrow 0 \rightarrow \cdots, \end{aligned}$$

which gives us $\mathbb{X} = \mathbb{Z}_2$.

For $k = 3$ and $k \geq 5$ we have the following exact sequence:

$$\begin{aligned} \cdots \rightarrow \pi_4(SU(k)) \rightarrow \pi_4(SU(k)/SO(k)) \rightarrow \pi_3(SO(k)) \rightarrow \pi_3(SU(k)) \rightarrow \cdots \\ \cdots \rightarrow 0 \rightarrow \mathbb{X} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \cdots, \end{aligned}$$

which gives us $\mathbb{X} = 0$.

The $k = 4$ case is also special,

$$\begin{aligned} \cdots \rightarrow \pi_4(SU(4)) \rightarrow \pi_4(SU(4)/SO(4)) \rightarrow \pi_3(SO(4)) \rightarrow \pi_3(SU(4)) \rightarrow \cdots \\ \cdots \rightarrow 0 \rightarrow \mathbb{X} \rightarrow \mathbb{Z} \otimes \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \cdots, \end{aligned}$$

which gives us $\mathbb{X} = \mathbb{Z}$.

A.4. π_5

The $k = 2$ case is special, as usual,

$$\begin{aligned} \cdots \rightarrow \pi_5(SO(2)) \rightarrow \pi_5(SU(2)) \rightarrow \pi_5(SU(2)/SO(2)) \rightarrow \pi_4(SO(2)) \rightarrow \cdots \\ \cdots \rightarrow 0 \rightarrow \mathbb{Z}_2 \rightarrow \mathbb{X} \rightarrow 0 \rightarrow \cdots, \end{aligned}$$

which gives us $\mathbb{X} = \mathbb{Z}_2$.

For $k = 3, 5$ we have the following exact sequence:

$$\begin{aligned} \cdots \rightarrow \pi_5(SU(k)) \rightarrow \pi_5(SU(k)/SO(k)) \rightarrow \pi_4(SO(k)) \rightarrow \pi_4(SU(k)) \rightarrow \cdots \\ \cdots \rightarrow \mathbb{Z} \rightarrow \mathbb{X} \rightarrow \mathbb{Z}_2 \rightarrow 0 \rightarrow \cdots, \end{aligned}$$

implying two alternatives,

$$\mathbb{X} = \mathbb{Z} \quad \text{or} \quad \mathbb{X} = \mathbb{Z} \otimes \mathbb{Z}_2.$$

In [16] it is shown that the last option is the correct one.

The $k = 4$ case is distinct,

$$\begin{aligned} \cdots \rightarrow \pi_5(SU(4)) \rightarrow \pi_5(SU(4)/SO(4)) \rightarrow \pi_4(SO(4)) \rightarrow \pi_4(SU(4)) \rightarrow \cdots \\ \cdots \rightarrow \mathbb{Z} \rightarrow \mathbb{X} \rightarrow \mathbb{Z}_2 \otimes \mathbb{Z}_2 \rightarrow 0 \rightarrow \cdots, \end{aligned}$$

which gives us the alternatives

$$\mathbb{X} = \mathbb{Z} \otimes \mathbb{Z}_2 \quad \text{or} \quad \mathbb{X} = \mathbb{Z} \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2.$$

It was shown in [16] that the last choice is the correct one.

For $k = 6$ we get

$$\begin{aligned} \dots \rightarrow \pi_5(SO(k)) \rightarrow \pi_5(SU(k)) \rightarrow \pi_5(SU(k)/SO(k)) \rightarrow \pi_4(SO(k)) \rightarrow \dots \\ \dots \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{X} \rightarrow 0 \rightarrow \dots, \end{aligned}$$

which is not enough to find \mathbb{X} . In [16] it was shown that $\mathbb{X} = \mathbb{Z}$.

For $k > 6$:

$$\begin{aligned} \dots \rightarrow \pi_5(SO(k)) \rightarrow \pi_5(SU(k)) \rightarrow \pi_5(SU(k)/SO(k)) \rightarrow \pi_4(SO(k)) \rightarrow \dots \\ \dots \rightarrow 0 \rightarrow \mathbb{Z} \rightarrow \mathbb{X} \rightarrow 0 \rightarrow \dots, \end{aligned}$$

which gives $\mathbb{X} = \mathbb{Z}$.

Appendix B. Four-derivatives term

The four-derivatives term can be computed from equation (6). Let us introduce the following compact notation:

$$S_{(\theta,\eta)}^{\mu\nu} = \partial_\mu \theta \partial_\nu \eta - \partial_\mu \eta \partial_\nu \theta, \tag{B.1}$$

and the same for all other possible coordinate pairings among $\theta, \eta, \alpha, \beta, \gamma$. Then we obtain, after a rather straightforward but quite cumbersome calculation, the following explicit expression:

$$\begin{aligned} \mathcal{L}_4 = & 8 \sin^2 \theta (S_{(\theta,\alpha)}^{\mu\nu})^2 + (1 - \cos \sqrt{3} \eta \cos \theta - \cos \alpha \sin \sqrt{3} \eta \sin \theta) (S_{(\theta,\beta)}^{\mu\nu})^2 \\ & + \sin^2 \theta (1 - \cos \sqrt{3} \eta \cos \theta - \cos \alpha \sin \sqrt{3} \eta \sin \theta) (S_{(\alpha,\beta)}^{\mu\nu})^2 \\ & + 3(1 - \cos \sqrt{3} \eta \cos \theta - \cos \alpha \sin \sqrt{3} \eta \sin \theta) (S_{(\eta,\beta)}^{\mu\nu})^2 \\ & + 3 \sin^2 \frac{\beta}{2} (1 - \cos \sqrt{3} \eta \cos \theta + \cos \alpha \sin \sqrt{3} \eta \sin \theta) (S_{(\gamma,\eta)}^{\mu\nu})^2 \\ & + \frac{1}{2} [8 \sin^2 \theta + (1 - \cos \sqrt{3} \eta \cos \theta + \cos \alpha \sin \sqrt{3} \eta \sin \theta) \\ & + \cos \beta (8 \sin^2 \theta - (1 - \cos \sqrt{3} \eta \cos \theta + \cos \alpha \sin \sqrt{3} \eta \sin \theta))] (S_{(\theta,\gamma)}^{\mu\nu})^2 \\ & + \sin^2 \theta \sin^2 \frac{\beta}{2} (1 - \cos \sqrt{3} \eta \cos \theta + \cos \alpha \sin \sqrt{3} \eta \sin \theta) (S_{(\alpha,\gamma)}^{\mu\nu})^2 \\ & + \frac{1}{8} [4 - \cos \alpha \sin \sqrt{3} \eta \sin^3 \theta - \cos \alpha \cos \beta \sin \sqrt{3} \eta \sin^3 \theta + 3 \cos \beta \sin^2 \theta \\ & + \sin^2 \theta - 3 \cos \alpha \sin \sqrt{3} \eta \sin \theta - 3 \cos \alpha \cos \beta \sin \sqrt{3} \eta \sin \theta \\ & - (\cos \beta - 1) \cos^2 \sqrt{3} \eta + \cos \beta \sin^2 \sqrt{3} \eta - \sin^2 \sqrt{3} \eta \\ & - \cos \sqrt{3} \eta \cos \theta \left(6 - 2 \cos \beta - 4 \cos^2 \frac{\beta}{2} \cos 2\theta \right) \\ & - \cos^2 \theta (1 - 3 \cos \alpha \sin \sqrt{3} \eta \sin \theta + \cos \beta (3 - 3 \cos \alpha \sin \sqrt{3} \eta \sin \theta))] \\ & \times (S_{(\beta,\gamma)}^{\mu\nu})^2 - 2 \sin \alpha \sin \frac{\beta}{2} \sin \sqrt{3} \eta \sin^3 \theta (S_{(\beta,\alpha)}^{\mu\nu}) (S_{(\gamma,\alpha)}^{\mu\nu}) \\ & + 2 \cos \frac{\beta}{2} \sin^2 \theta (1 - \cos \sqrt{3} \eta \cos \theta - \cos \alpha \sin \sqrt{3} \eta \sin \theta) (S_{(\beta,\alpha)}^{\mu\nu}) (S_{(\beta,\gamma)}^{\mu\nu}) \\ & + \sin \alpha \sin \beta \sin \sqrt{3} \eta \sin^3 \theta (S_{(\gamma,\alpha)}^{\mu\nu}) (S_{(\gamma,\beta)}^{\mu\nu}) \\ & - 2\sqrt{3} \cos \frac{\beta}{2} \sin \alpha \sin \theta (\cos \sqrt{3} \eta - \cos \theta) (S_{(\gamma,\beta)}^{\mu\nu}) (S_{(\eta,\beta)}^{\mu\nu}) \end{aligned}$$

$$\begin{aligned}
& -\sqrt{3} \cos \alpha \sin \beta \sin \theta (\cos \sqrt{3} \eta - \cos \theta) (S_{(\gamma, \beta)}^{\mu\nu}) (S_{(\gamma, \eta)}^{\mu\nu}) \\
& -6 \sin \alpha \sin \frac{\beta}{2} \sin \sqrt{3} \eta \sin \theta (S_{(\eta, \beta)}^{\mu\nu}) (S_{(\eta, \gamma)}^{\mu\nu}) \\
& +2\sqrt{3} (\cos \sqrt{3} \eta - \cos \theta) \sin \alpha \sin^2 \frac{\beta}{2} \sin \theta (S_{(\gamma, \alpha)}^{\mu\nu}) (S_{(\gamma, \eta)}^{\mu\nu}) \\
& -2\sqrt{3} (\cos \sqrt{3} \eta - \cos \theta) \sin \alpha \sin \theta (S_{(\beta, \alpha)}^{\mu\nu}) (S_{(\beta, \eta)}^{\mu\nu}) \\
& -3 (\cos \sqrt{3} \eta - \cos \theta) \sin \beta \sin \theta (S_{(\gamma, \beta)}^{\mu\nu}) (S_{(\gamma, \theta)}^{\mu\nu}) \\
& -2 \sin \alpha \sin \frac{\beta}{2} \sin \sqrt{3} \eta \sin \theta (S_{(\theta, \beta)}^{\mu\nu}) (S_{(\theta, \gamma)}^{\mu\nu}) \\
& +16 \cos \frac{\beta}{2} \sin^2 \theta (S_{(\theta, \alpha)}^{\mu\nu}) (S_{(\theta, \gamma)}^{\mu\nu}) \\
& -2\sqrt{3} \sin^2 \frac{\beta}{2} (\cos \alpha (\cos \sqrt{3} \eta \cos \theta - 1) - \sin \sqrt{3} \eta \sin \theta) (S_{(\eta, \gamma)}^{\mu\nu}) (S_{(\theta, \gamma)}^{\mu\nu}) \\
& +2\sqrt{3} (\cos \alpha (\cos \sqrt{3} \eta \cos \theta - 1) + \sin \sqrt{3} \eta \sin \theta) (S_{(\eta, \beta)}^{\mu\nu}) (S_{(\theta, \beta)}^{\mu\nu}) \\
& +4\sqrt{3} (\cos \sqrt{3} \eta \cos \theta - 1) \sin \alpha \sin \frac{\beta}{2} (S_{(\eta, \gamma)}^{\mu\nu}) (S_{(\theta, \beta)}^{\mu\nu}) \\
& +2\sqrt{3} (\cos \sqrt{3} \eta \cos \theta - 1) \sin \alpha \sin \frac{\beta}{2} (S_{(\gamma, \beta)}^{\mu\nu}) (S_{(\theta, \eta)}^{\mu\nu}) \\
& -6 (\cos \theta - \cos \sqrt{3} \eta) \sin \frac{\beta}{2} \sin \theta (S_{(\gamma, \beta)}^{\mu\nu}) (S_{(\theta, \alpha)}^{\mu\nu}) \\
& -4\sqrt{3} \sin \theta (\cos \sqrt{3} \eta - \cos \theta) \cos \alpha \sin \frac{\beta}{2} (S_{(\gamma, \alpha)}^{\mu\nu}) (S_{(\eta, \beta)}^{\mu\nu}) \\
& -2\sqrt{3} \sin \theta (\cos \sqrt{3} \eta - \cos \theta) \cos \alpha \sin \frac{\beta}{2} (S_{(\gamma, \beta)}^{\mu\nu}) (S_{(\alpha, \eta)}^{\mu\nu}). \tag{B.2}
\end{aligned}$$

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